THE CONNECTEDNESS OF THE MODULI SPACE OF MAPS TO HOMOGENEOUS SPACES

B. KIM AND R. PANDHARIPANDE

0. Introduction

Let X be a compact algebraic homogeneous space: $X = \mathbf{G}/\mathbf{P}$ where $\mathbf{G}$ is a connected complex semisimple algebraic group and $\mathbf{P}$ is a parabolic subgroup. Let $\beta \in H_2(X, \mathbb{Z})$. The (coarse) moduli space $\overline{M}_{g,n}(X,\beta)$ of n-pointed genus g stable maps parameterizes the data

$$[\mu : C \to X, p_1, \ldots , p_n]$$

satisfying:

(i) C is a complex, projective, connected, reduced, (at worst) nodal curve of arithmetic genus g.
(ii) The points $p_i \in C$ are distinct and lie in the nonsingular locus.
(iii) $\mu_*[C] = \beta$.
(iv) The pointed map μ has no infinitesimal automorphisms.

Since X is convex, the genus 0 moduli space $\overline{M}_{0,n}(X,\beta)$ is of pure dimension

$$\dim(X) + \int_\beta c_1(T_X) + n - 3.$$

Moreover, $\overline{M}_{0,n}(X,\beta)$ is locally the quotient of a nonsingular variety by a finite group. For general g, the space $\overline{M}_{g,n}(X,\beta)$ may have singular components of different dimensions. Stable maps in algebraic geometry were first defined in [Ko]. Basic properties of the moduli space $\overline{M}_{g,n}(X,\beta)$ can be found in [BM], [FP], and [KoM]. The following connectedness result is proven here.

Theorem 1. *$\overline{M}_{g,n}(\mathbf{G}/\mathbf{P},\beta)$ is a connected variety.*

This result may be viewed as analogous to the connectedness of the Hilbert scheme of projective space proven by Hartshorne. As in [Har], connectedness is obtained via maximal degenerations.

Since $\overline{M}_{0,n}(X,\beta)$ has quotient singularities, connectedness is equivalent to irreducibility.

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Corollary 1. $\overline{M}_{0,n}(\mathbf{G}/\mathbf{P}, \beta)$ *is an irreducible variety.*

Corollary 1 is easy to verify in case X is a projective space. When X is a Grassmannian, the irreducibility follows from Strømme's Quot scheme analysis [S]. A proof of Corollary 1 can be found in case $\mathbf{G} = \mathbf{SL}$ in [MM]. For the variety of partial flags in $\mathbb{C}^n$, a proof of irreducibility using flag-Quot schemes is established in [Ki]. Results of Harder closely related to Corollary 1 appear in [Ha]. There is an independent proof by J. Thomsen for the irreducibility of $\overline{M}_{0,n}(\mathbf{G}/\mathbf{P}, \beta)$ in [T].

The moduli space $\overline{M}_{g,n}(X, \beta)$ has a natural locally closed decomposition indexed by stable, pointed, modular graphs τ (see [BM]). The strata correspond to maps with domain curves of a fixed topological type and a fixed distribution β_τ of β. The graph τ determines a complete moduli space of stable maps

$$\overline{M}_{\tau,n}(X, \beta_\tau)$$

together with a canonical morphism:

$$\pi_\tau : \overline{M}_{\tau,n}(X, \beta_\tau) \to \overline{M}_{g,n}(X, \beta). \tag{1}$$

A closed decomposition is determined by the images of these morphisms (1). Theorem 1 is a special case of the following result.

Theorem 2. $\overline{M}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau)$ *is a connected variety.*

Since $\overline{M}_{\tau,n}(X, \beta_\tau)$ is normal in the genus 0 case, we obtain the corresponding corollary.

Corollary 2. *Let $g = 0$. $\overline{M}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau)$ is an irreducible variety.*

In particular, all the boundary divisors of $\overline{M}_{0,n}(X, \beta)$ are irreducible.

Theorem 2 is proven by studying the maximal torus action on X. The method is to degenerate a general $\mathbf{G}$-translate of a map $\mu : C \to X$ onto a canonical 1-dimensional configuration of $\mathbb{P}^1$'s in X determined by the maximal torus and the Bialynicki-Birula stratification of X.

In the genus 0 case, we study the Bialynicki-Birula stratification of $\overline{M}_{0,n}(X, \beta)$. The following result is then deduced from the rationality of torus fixed components.

Theorem 3. $\overline{M}_{0,n}(\mathbf{G}/\mathbf{P}, \beta)$ *is rational.*

The fixed component rationality is equivalent to a rationality result for certain quotients of $\mathbf{SL}_2$-representations proven by Katsylo and Bogomolov [Ka], [Bog]. It should be noted that the fixed components will in general be contained in the boundary of the moduli space of maps –

the compactifaction by stable maps therefore plays an important role in the proof.

The rationality of the Hilbert schemes of rational curves in projective space (birational to $\overline{M}_{0,0}(\mathbb{P}^r, d)$) is a consequence of Katsylo's results [Ka] and was also studied by Hirschowitz in [Hi].

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1. The torus action on $\mathbf{G}/\mathbf{P}$

Let $\mathbf{G}$ be a connected complex semisimple algebraic group. Let $\mathbf{P}$ be a parabolic subgroup. Select a maximal algebraic torus $\mathbf{T}$ and Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ satisfying:

$$\mathbf{T} \subset \mathbf{B} \subset \mathbf{P} \subset \mathbf{G}.$$

Let $(\mathbf{G}/\mathbf{P})^{\mathbf{T}}$ denote the fixed point set of the left $\mathbf{T}$-action on $\mathbf{G}/\mathbf{P}$. Three special properties of this $\mathbf{T}$-action will be needed:

(i) The $\mathbf{T}$-action has isolated fixed points.
(ii) For every point $p \in (\mathbf{G}/\mathbf{P})^{\mathbf{T}}$, there exits a $\mathbf{T}$-invariant open set U_p containing p which is $\mathbf{T}$-equivalent to a vector space representation of $\mathbf{T}$.
(iii) Let $\mathbb{C}^* \subset \mathbf{T}$ correspond to an interior point of a Weyl chamber. Then, $(\mathbf{G}/\mathbf{P})^{\mathbb{C}^*} = (\mathbf{G}/\mathbf{P})^{\mathbf{T}}$, and the Bialynicki-Birula decomposition obtained from the $\mathbb{C}^*$-action is an affine *stratification* of $\mathbf{G}/\mathbf{P}$.

A stratification is a decomposition such that the closures of the strata are unions of strata. In general, the Bialynicki-Birula decomposition obtained from a $\mathbb{C}^*$-action on a nonsingular variety need not be a stratification.

The claims (i)-(iii) are well known. Only a brief summary of the arguments will be presented here. Let $\mathbf{W}$ be the Weyl group of $\mathbf{G}$ relative to $\mathbf{T}$.

Lemma 1. $|(\mathbf{G}/\mathbf{B})^{\mathbf{T}}| = |\mathbf{W}|$, *and* $\mathbf{W}$ *acts transitively on* $(\mathbf{G}/\mathbf{B})^{\mathbf{T}}$.

Proof. See, for example, [Bor]. $\square$

In particular, $(\mathbf{G}/\mathbf{B})^{\mathbf{T}}$ is a finite set.

Lemma 2. *The natural map* $(\mathbf{G}/\mathbf{B})^{\mathbf{T}} \to (\mathbf{G}/\mathbf{P})^{\mathbf{T}}$ *is surjective.*

Proof. Let $p \in (\mathbf{G}/\mathbf{P})^{\mathbf{T}}$. The invariant fiber (isomorphic to $\mathbf{P}/\mathbf{B}$) over the fixed point p is a nonsingular projective variety, and hence contains a $\mathbf{T}$-fixed point by the Borel fixed point theorem (or, alternatively, this is a Hamiltonian action on a compact manifold). □

Therefore, $\mathbf{W}$ acts transitively on the finite set $(\mathbf{G}/\mathbf{P})^{\mathbf{T}}$.

A representation $\psi : \mathbf{T} \to \mathbf{GL}(V)$ is *fully definite* if there exists a $\mathbb{C}^*$-basis of $\mathbf{T}$ for which all the weights of the representation are positive integers. Equivalently, a fully definite representation can be written

$$\psi(t_1, \dots, t_r)v_j = \prod_{i=1}^{r} t_i^{\lambda_{ij}} \cdot v_j$$

where $\lambda_{ij} > 0$ for some choice of $\mathbb{C}^*$-basis of $\mathbf{T}$ and $\mathbb{C}$-basis $\{v_j\}$ of V.

The point $1 \in \mathbf{G}/\mathbf{B}$ corresponding to the identity element of $\mathbf{G}$ is a $\mathbf{T}$-fixed point. The $\mathbf{T}$-action induces a representation

$$\phi : \mathbf{T} \to \mathbf{GL}(\mathrm{Tan}_1 \mathbf{G}/\mathbf{B}).$$

Lemma 3. *The representation* ϕ *is fully definite.*

Proof. The natural quotient map $q : \mathbf{G} \to \mathbf{G}/\mathbf{B}$ is $\mathbf{T}$-equivariant for the conjugation action on $\mathbf{G}$ and the left action on $\mathbf{G}/\mathbf{B}$. The differential of q yields an isomorphism from the Adjoint representation of $\mathbf{T}$ on $\mathrm{Lie}(\mathbf{G})/\mathrm{Lie}(\mathbf{B})$ to ϕ. $\mathrm{Lie}(\mathbf{G})/\mathrm{Lie}(\mathbf{B})$ is the space of positive roots. This $\mathbf{T}$-representation space has n simple roots (where n is the rank of $\mathbf{G}$). All the 1-dimensional representations in $\mathrm{Lie}(\mathbf{G})/\mathrm{Lie}(\mathbf{B})$ are non-negative tensor products of these simple roots. Moreover, the n weight vectors of these simple roots are independent in the lattice of 1-dimensional representations of the torus $\mathbf{T}$. Lemma 3 now follows from Lemma 4 below. □

Lemma 4. *Let* $\psi : \mathbf{T} \to \mathbf{GL}(\mathbb{C}^n)$ *be an* n *dimensional representation of a rank* n *torus* $\mathbf{T}$*. If the* $n \times n$ *matrix of weights is nonsingular, then the representation is fully definite.*

Proof. See [Bi]. □

Lemma 5. *The* $\mathbf{T}$*-representation* $\mathrm{Tan}_1 \mathbf{G}/\mathbf{P}$ *is fully definite.*

Proof. There is a surjection of $\mathbf{T}$-modules given by the differential $\mathrm{Tan}_1 \mathbf{G}/\mathbf{B} \to \mathrm{Tan}_1 \mathbf{G}/\mathbf{P}$.

□

Proposition 1. *For every $p \in (\mathbf{G}/\mathbf{P})^{\mathbf{T}}$, there exists a $\mathbf{T}$-invariant Zariski open set $U_p \subset \mathbf{G}/\mathbf{P}$ of p which is $\mathbf{T}$-equivalent to a vector space representation of $\mathbf{T}$.*

Proof. By a theorem of Bialynicki-Birula [Bi], it suffices to show the tangent representation of $\mathbf{T}$ is fully definite at p. This is a consequence of Lemma 5 and the transitivity of the $\mathbf{W}$-action on $(\mathbf{G}/\mathbf{P})^{\mathbf{T}}$. (In fact, only definiteness of the tangent representation is needed in [Bi].) $\square$

Let $\mathbb{C}^* \subset \mathbf{T}$ correspond to an interior point of a Weyl chamber. By the analysis of the tangent representation ϕ, every point of $(\mathbf{G}/\mathbf{B})^{\mathbf{T}}$ is an isolated fixed point of $\mathbb{C}^*$. The equality $(\mathbf{G}/\mathbf{B})^{\mathbb{C}^*} = (\mathbf{G}/\mathbf{B})^{\mathbf{T}}$ follows. Since the map $(\mathbf{G}/\mathbf{B})^{\mathbb{C}^*} \to (\mathbf{G}/\mathbf{P})^{\mathbb{C}^*}$ is surjective, $(\mathbf{G}/\mathbf{P})^{\mathbb{C}^*} = (\mathbf{G}/\mathbf{P})^{\mathbf{T}}$.

For each $p \in (\mathbf{G}/\mathbf{P})^{\mathbf{T}}$, let A_p be the set of points $x \in \mathbf{G}/\mathbf{P}$ such that

$$\lim_{t \to 0} tx = p.$$

By Proposition 1, A_p is isomorphic to the affine space $\mathbb{C}^{r_p}$ where r_p is the number of positive weights in the $\mathbb{C}^*$-representation $\mathrm{Tan}_p\mathbf{G}/\mathbf{P}$. The set $\{A_p\}$ is the Bialynicki-Birula affine decomposition of $\mathbf{G}/\mathbf{P}$. In fact, $\{A_p\}$ coincides (up to the Weyl group action) with the (open) Schubert cell stratification of $\mathbf{G}/\mathbf{P}$. This is essentially proven in [Bor] for the case $\mathbf{G}/\mathbf{B}$. The general case $\mathbf{G}/\mathbf{P}$ is proven in [A]. Therefore, $\{A_p\}$ is a stratification.

2. The $\mathbb{C}^*$-flow

Let $\mathbb{C}^* \subset \mathbf{T}$ correspond to an interior point of a Weyl chamber. Let $s, x_1, \dots, x_l \in (\mathbf{G}/\mathbf{P})^{\mathbf{T}}$ be the fixed points corresponding to the unique maximal dimensional stratum A_s and the complete set of codimension 1 strata, $A_1, \dots, A_l$, respectively. The points of A_s flow ($t \to 0$) to s, and the points of A_i flow ($t \to 0$) to x_i. Let $U = A_s \cup A_1 \cup \ldots \cup A_l$. Since the Bialynicki-Birula decomposition $\{A_p\}$ is a stratification, U is a Zariski open set with complement of codimension at least 2.

The inverse action of $\mathbb{C}^*$ on $\mathbf{G}/\mathbf{P}$ is also a torus action on $\mathbf{G}/\mathbf{P}$ with the same fixed point set. Let $A'_s, A'_1, \dots, A'_l$ be the affine strata for the inverse action corresponding to the fixed points $s, x_1, \dots, x_l$. Let $\dim(\mathbf{G}/\mathbf{P}) = m$. Since,

$$\dim(A_p) + \dim(A'_p) = m,$$

$A'_1, \cdots, A'_l$ are the complete set of 1-dimensional strata for the inverse action. Moreover, the closure $P_i = \overline{A'}_i$ can contain only the unique 0-dimensional stratum $A'_s = s$. We have shown the closures P_i are contained in U. Each P_i is isomorphic to $\mathbb{P}^1$ (Chevelley [C] proves the

closed Schubert cells have singularities in codimension at least 2). The intersection pairing

$$P_i \cap \overline{A}_j = \delta(i-j)$$

follows from the above analysis. Since the closed strata of the inverse action freely generate the integral homology, the classes

$$[P_1], \ldots, [P_l] \in H_2(\mathbf{G}/\mathbf{P}, \mathbb{Z})$$

span an integral basis of $H_2(\mathbf{G}/\mathbf{P}, \mathbb{Z})$.

Let $f : C \to \mathbf{G}/\mathbf{P}$ be a non-constant stable map satisfying the following properties:

(i) The image $f(C)$ lies in U.

(ii) C intersects (via f) the divisors A_i transversely at nonsingular points of C.

(iii) All the markings of C have image in A_s.

If $[f]$ represents the class

$$\beta = \sum_{i=1}^{l} a_i [P_i] \in H_2(\mathbf{G}/\mathbf{P}, \mathbb{Z}),$$

then let C meet A_i at the a_i distinct points

$$\{x_{i,1}, \ldots, x_{i,a_i}\}.$$

We will study the induced $\mathbb{C}^*$-action on $\overline{M}_{g,n}(\mathbf{G}/\mathbf{P}, \beta)$ by translation of maps. Let $F : C_0 \to \mathbf{G}/\mathbf{P}$ be the limit in the space of stable maps,

$$F = \lim_{t \to 0} \ tf$$

where $t \in \mathbb{C}^*$.

Define a map $\tilde{F} : \tilde{C} \to \mathbf{G}/\mathbf{P}$ as follows. Let the domain $\tilde{C}$ be:

$$\tilde{C} = C \ \cup \ \bigcup_{i=1}^{l} \left(\cup_{j=1}^{a_i} \mathbf{P}^1_{i,j} \right)$$

where $\mathbf{P}^1_{i,j}$ is a projective line attached to C at the point $x_{i,j}$. Let the markings of $\tilde{C}$ coincide with the markings of C (note the markings of C are disjoint from the set $\{x_{i,j}\}$ by condition (ii)). Define $\tilde{F}$ by $\tilde{F}(C \subset \tilde{C}) = s$ and

$$\tilde{F}|_{\mathbf{P}^1_{i,j}} : \mathbf{P}^1_{i,j} \stackrel{\sim}{=} P_i$$

for each i and j.

Proposition 2. *If f satisfies conditions (i-iii), then the $t \to 0$ limit F equals the stabilization of $\tilde{F}$.*

Proof. Let $\triangle^\circ \subset \triangle$ be the punctured holomorphic disk at the origin. Let

$$h : C \times \triangle^\circ \to \mathbf{G}/\mathbf{P}$$

be the map defined by $h(c,t) = tf(c)$. The $\mathbb{C}^*$-action on A_s extends to a map

$$\mathbb{C} \times A_s \to A_s$$

since the $\mathbb{C}^*$-action on A_s is a vector space representation with positive weights. The map h thus extends to a map

$$h : C \times \triangle \setminus \{x_{i,j} \times 0\} \to \mathbf{G}/\mathbf{P}$$

since the f-image of $C \setminus \{x_{i,j}\}$ lies in A_s. Note,

$$h(C \setminus \{x_{i,j}\}, 0) = s. \tag{2}$$

After a suitable blow-up

$$\gamma : S \to C \times \triangle$$

supported along the isolated nonsingular points $\{x_{i,j} \times 0\}$ of $C \times \triangle$, there is a morphism $h' : S \to \mathbf{G}/\mathbf{P}$.

The limit as $t \to 0$ of $tf(x_{i,j})$ equals x_i. Hence, the exceptional divisor $C_{i,j}$ of γ over $x_{i,j}$ connects the points x_i to s under the map h'. The image $h'(C_{i,j})$ thus represents an effective curve class containing the class $[P_i]$. By degree considerations over all the exceptional divisors $C_{i,j}$, we conclude $h'(C_{i,j})$ is of curve class exactly $[P_i]$. As P_i is the unique $\mathbb{C}^*$-fixed curve of class $[P_i]$ connecting the points x_i and s,

$$h'(C_{i,j}) = P_i.$$

We may assume S to be nonsingular (away from the original nodes of C) and each $C_{i,j}$ to be a normal crossings divisor – possibly after further blow-ups and base changes altering only the special fiber over $0 \in \triangle$. We then conclude each $C_{i,j}$ has a single component which is mapped to P_i isomorphically (and the other components of $C_{i,j}$ are contracted).

After blowing-down the h'-contracted components of each $C_{i,j}$, we obtain a map $h'' : S'' \to \mathbf{G}/\mathbf{P}$ which is a family of nodal maps over $\triangle$. The fiber of S'' over $t = 0$ is isomorphic to $\tilde{C}$. Moreover, the condition $\tilde{F}(C \subset \tilde{C}) = s$ follows directly from (2).

The limit stable map F is then simply obtained by stabilizing the map $\tilde{F}$. We have carried out the stable reduction of the family of maps tf (see [FP]). □

3. Connectedness

Let $[\mu]$ denote the point $[\mu : C \to X, p_1, \dots, p_n] \in \overline{M}_{g,n}(X, \beta)$. The stable, pointed, modular graph τ with $H_2(X, \mathbb{Z})$-structure canonically associated to $[\mu]$ consists of the following data:

- (i) The pointed dual graph of C:
  - (a) The vertices V_τ correspond to the irreducible components of the curve C.
  - (b) The edges correspond to the nodes.
  - (c) The markings correspond to the marked points p_i.
- (ii) The genus function, $g_\tau : V_\tau \to \mathbb{Z}^{\geq 0}$, where $g_\tau(v)$ is the geometric genus of the corresponding component of C.
- (iii) The $H_2(X, \mathbb{Z})$-structure, $\beta_\tau : V_\tau \to H_2(X, \beta)$, where $\beta_\tau(v)$ equals the μ push-forward of the fundamental class of the corresponding component of C.

Following [BM], define $M_{\tau,n}(X, \beta_\tau)$ to be the moduli space of maps μ *together* with an isomorphism of τ_μ with a fixed stable graph τ. The space $\overline{M}_{\tau,n}(X, \beta_\tau)$ is the compactification via stable maps where the vertices of V_τ may correspond to nodal curves. Note $M_{\tau,n}(X, \beta_\tau)$ may not be dense in $\overline{M}_{\tau,n}(X, \beta_\tau)$.

There is a canonical morphism

$$\pi_\tau : \overline{M}_{\tau,n}(X, \beta_\tau) \to \overline{M}_{g,n}(X, \beta).$$

As τ varies over possible graphs, the images of π_τ determine a (closed) decomposition of the moduli space of maps.

Let τ be a stable, pointed, modular graph with $H_2(\mathbf{G}/\mathbf{P}, \mathbb{Z})$-structure. The connectedness of $\overline{M}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau)$ will now be established.

Proof of Theorem 2. If $\beta_\tau = 0$, the irreducibility of $\overline{M}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau)$ is a direct consequence of the irreducibility of the corresponding stratum in $\overline{M}_{g,n}$ and the irreducibility of $\mathbf{G}/\mathbf{P}$. We may thus assume $\beta_\tau \neq 0$.

Fix the $\mathbb{C}^*$-action on $\mathbf{G}/\mathbf{P}$ as studied in Section 2. Consider an arbitrary point

$$[\mu] \in \overline{M}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau).$$

By the Kleiman-Bertini Theorem, a general $\mathbf{G}$-translate f of μ satisfies conditions (i-iii) of Section 2. As $\mathbf{G}$ is connected, $[\mu]$ is connected to its general $\mathbf{G}$-translate $[f]$.

The point $[f]$ is connected to the limit:

$$[F] = \lim_{t \to 0} [tf].$$

To prove the connectedness of $\overline{M}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau)$, it suffices to prove the set of limits F lies in a connected locus of the moduli space. We will first construct the required connected locus of $\overline{M}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau)$.

The pair (τ, β_τ) canonically determines a family of maps γ_b with nodal domains over a base $b \in B$. For $v \in V_\tau$, let $\beta_\tau(v) = \sum_i a_i^v [P_i]$. Define the base space B as follows:

$$B = \prod_{v \in V_\tau} \overline{M}_{g(v), \mathrm{val}(v) + \sum_i a_i^v},$$

where $\mathrm{val}(v)$ is the valence of v in τ (including nodes and markings). The extra $\sum_i a_i^v$ markings each correspond to a basis homology element – with a_j^v of these markings corresponding to $[P_j]$. The degenerate cases $\overline{M}_{0,1}$ and $\overline{M}_{0,2}$ in the product B are taken to be points. B is irreducible and hence connected.

For $b = \prod_v [b_v] \in B$, let

$$\gamma_b : D_b \to \mathbf{G}/\mathbf{P}$$

be defined as follows:

(i) D_b is obtained by attaching the curves b_v by connecting nodes as specified by τ and further attaching $\mathbb{P}^1$'s to each of the extra points $\sum_i a_i^v$.
(ii) For each subcurve $b_v \subset D_b$, $\gamma_b(b_v) = s$.
(iii) For each $\mathbb{P}^1$ corresponding to $[P_j]$, $\gamma_b(\mathbb{P}^1) \stackrel{\sim}{=} P_j$.

The family of maps γ_b over B then defines a morphism (via stabilization):

$$\epsilon : B \to \overline{M}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau).$$

Certainly the image variety $\epsilon(B)$ is connected.

By Proposition 2, the limit F is simply the stabilization of $[\tilde{F}]$. Since $\tilde{F} = \gamma_b$ for some b, the set of limits F lies in a connected locus of $\overline{M}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau)$. This concludes the proof of Theorem 2. $\square$

Theorem 1 is a special case of Theorem 2 (where τ has a single vertex). Corollary 2 is a simple consequences of Theorem 2.

Proof of Corollary 2. In the genus 0 case, τ is a tree with genus function identically zero. The moduli stack

$$\overline{\mathcal{M}}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau) \tag{3}$$

is constructed as a fiber product over the evaluation maps obtained from the edges of τ. We will prove $\overline{\mathcal{M}}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau)$ is a nonsingular Deligne-Mumford stack by induction on the number of vertices of τ.

First, suppose τ has only 1 vertex v. Then, the moduli stack (3) is $\overline{\mathcal{M}}_{0,\mathrm{val}(v)}(\mathbf{G}/\mathbf{P}, \beta_\tau(v))$ – a nonsingular moduli stack by the convexity of $\mathbf{G}/\mathbf{P}$.

Next, let τ have m vertices and let v be an extremal vertex (v is incident to exactly 1 edge). Let $p \in \mathbf{G}/\mathbf{P}$ be a point. By the Kleiman-Bertini Theorem,

$$\mathrm{ev}_1^{-1}(p) \subset \overline{\mathcal{M}}_{0,\mathrm{val}(v)}(\mathbf{G}/\mathbf{P}, \beta_\tau(v)) \tag{4}$$

is a nonsingular Deligne-Mumford stack for the general point p (and hence *every* point p). Let τ' be the graph obtained by removing v from τ and adding an extra marking corresponding to the broken node. The moduli stack (3) is fibered over

$$\overline{\mathcal{M}}_{\tau',n'+1}(\mathbf{G}/\mathbf{P}, \beta'_\tau) \tag{5}$$

with fiber (4). As (5) is nonsingular by induction, the stack (3) is thus nonsingular. This completes the induction step.

Finally, since $\overline{\mathcal{M}}_{\tau,n}(\mathbf{G}/\mathbf{P}, \beta_\tau)$ is a nonsingular and connected Deligne-Mumford stack, it is irreducible. $\square$

4. Rationality

We first review a basic rationality result proven in a sequence papers by Katsylo and Bogomolov [Ka], [Bog]. Let $V = \mathbb{C}^2$ be a vector space. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive integers with $\sum_i a_i \geq 3$. Then, the quotient

$$\mathbb{P}(\mathrm{Sym}^{a_1} V^*) \times \cdots \times \mathbb{P}(\mathrm{Sym}^{a_n} V^*) \; // \; \mathbf{PGL}(V) \tag{6}$$

is a rational variety – we may take any non-empty invariant theory quotient. Geometrically, the quotient (6) is birational to the moduli space quotient

$$M_{0,\sum_i a_i} \; / \; \Sigma_{a_1} \times \Sigma_{a_2} \times \cdots \times \Sigma_{a_n} \tag{7}$$

where Σ is the symmetric group. Essentially, the rationality of (6) is deduced from rationality in case $n = 1$ [Ka]. Proofs in the $n = 1$ case may be found in [Ka], [Bog].

We will also need the following simple Lemma.

Lemma 6. *Let W be any finite dimensional linear representation of $\mathbf{A}$ where $\mathbf{A} = \Sigma_2$ or $\mathbf{A} = \Sigma_3$. Then, $W/\mathbf{A}$ is rational.*

Proof. By the complete reducibility of representations and the fact that a $\mathbf{GL}$-bundle is locally trivial in the Zariski topology, it suffices to prove the Lemma in case W is an irreducible representation. It is then

easily checked by hand the two irreducible representation of Σ_2 and the three irreducible representations of Σ_3 have rational quotients. $\square$

Proof of Theorem 3. Fix the $\mathbb{C}^*$-action on $\mathbf{G}/\mathbf{P}$ as studied in Section 2. We first consider the moduli space

$$\overline{M} = \overline{M}_{0,n}(\mathbf{G}/\mathbf{P}, \beta = \sum_i a_i[P_i])$$

where the property

$$n + \sum_i a_i \geq 4 \tag{8}$$

is satisfied.

Let τ be the graph with a single vertex v with n markings, and let $\beta_\tau(v) = \sum_i a_i[P_i]$. Let γ_b over B be the family of maps constructed canonically from (τ, β_τ) in the proof of Theorem 2. The base B is simply:

$$B = \overline{M}_{0,n+\sum_i a_i}. \tag{9}$$

The map γ_b over a general point $b \in B$ has no map automorphisms (as $n + \sum_i a_i \geq 4$). Hence, the image $\epsilon(B)$ in $\overline{M}$ intersects the nonsingular (automorphism-free) locus of the moduli space $\overline{M}^0 \subset \overline{M}$. Let

$$\epsilon(B)^0 = \epsilon(B) \cap \overline{M}^0,$$

and let $B^0 = \epsilon^{-1}(\epsilon(B)^0)$. The map

$$B^0 \to \epsilon(B)^0$$

is simply a quotient of B^0 by the natural $\Sigma_{a_1} \times \cdots \times \Sigma_{a_n}$ action on (9). By the rationality result (6), $\epsilon(B)^0$ is rational.

Consider now the $\mathbb{C}^*$-action on $\overline{M}^0$ by translation. As $\overline{M}^0$ is a nonsingular, irreducible, quasi-projective variety, we may study the Bialynicki-Birula stratification of $\overline{M}^0$. By the proof of Theorem 2, $\epsilon(B)^0$ is a $\mathbb{C}^*$-fixed locus which contains the limit,

$$\lim_{t\to 0} t[f],$$

of the general point $[f] \in \overline{M}^0$. By [Bi], $\overline{M}^0$ is birational to an affine bundle over $\epsilon(B)^0$. Therefore, $\overline{M}$ is rational. The proof of Theorem 3 is complete in case (8) is satisfied.

Next, we will consider the case where the sum (8) is at most 3. In this case, the base B is a point. If $\epsilon(B)$ lies in the automorphism-free locus, the previous argument proving the rationality of $\overline{M}_{0,n}(\mathbf{G}/\mathbf{P}, \beta)$

is still valid. There are exactly four cases in which the point $\epsilon(B)$ corresponds to a map with nontrivial automorphisms:

(i) $n = 0$, $\beta = 3[P_i]$.
(ii) $n = 0$, $\beta = 2[P_i] + [P_j]$, $i \neq j$.
(iii) $n = 0$, $\beta = 2[P_i]$.
(iv) $n = 1$, $\beta = 2[P_i]$.

Here, the Deligne-Mumford stack structure of these moduli spaces is important. The automorphism group in case (i) is Σ_3 and in cases (ii-iv) is Σ_2. In each case, we will show the coarse moduli space $\overline{M}_{0,n}(\mathbf{G}/\mathbf{P}, \beta)$ is birational to a quotient of a linear representation of the corresponding automorphism group.

Consider first the case (i): $n = 0$, $\beta = 3[P_i]$. Let $\epsilon(B) = [\gamma]$. Let $[\mu]$ denote the unique 3-pointed stable map obtained from γ by marking each $\mathbb{P}^1 \cong P_i$ by a point lying over x_i. Certainly, $[\mu] \in \overline{M}^0_{0,3}(\mathbf{G}/\mathbf{P}, \beta)$

We will study:

$$N \subset \overline{M}^0_{0,3}(\mathbf{G}/\mathbf{P}, \beta)$$

where N is the component of the locus of transverse intersection of the three divisors $\mathrm{ev}_1^{-1}(\overline{A}_i)$, $\mathrm{ev}_2^{-1}(\overline{A}_i)$, and $\mathrm{ev}_3^{-1}(\overline{A}_i)$ containing $[\mu]$. The torus $\mathbb{C}^*$ acts on N by translation. By an argument exactly parallel to the flow result of Proposition 2, we deduce

$$\lim_{t \to 0} t[f] = [\mu]$$

for a general element $[f] \in N$. As N is a nonsingular, quasi-projective scheme, Theorem 2.5 of [Bi] implies that N is $\mathbb{C}^*$-equivariantly birational to the tangent $\mathbb{C}^*$-representation at $[\mu]$.

There is a Σ_3-action on N by permutation of the markings. The $\mathbb{C}^*$ and Σ_3 actions commute. A slightly refined version of Theorem 2.5 of [Bi] shows N is $\mathbb{C}^* \times \Sigma_3$-equivariantly birational to the tangent $\mathbb{C}^* \times \Sigma_3$-representation at $[\mu]$. Lemma 7 below explains the refinements of the results of [Bi] needed here. N/Σ_3 is birational to $\overline{M}_{0,0}(\mathbf{G}/\mathbf{P}, \beta)$. Hence, by Lemma 6, Theorem 3 is proven in case (i).

A similar strategy is used in cases (ii-iv). In each of these cases, let $\epsilon(B) = [\gamma]$ and let $[\mu]$ denote the rigidification by adding 2 new markings $\bullet, \bullet'$ which lie over x_i. The locus N is chosen as the corresponding transverse intersection locus of $\mathrm{ev}_\bullet^{-1}(\overline{A}_i)$ and $\mathrm{ev}_{\bullet'}^{-2}(\overline{A}_i)$ in the maps space with the new markings. N is then $\mathbb{C}^* \times \Sigma_2$-equivariantly birational to the tangent $\mathbb{C}^* \times \Sigma_2$-representation of N at $[\mu]$ by the refined Lemma 7. Theorem 3 is then a consequence of Lemma 6 since N/Σ_2 is birational to the moduli space of maps considered in the case. $\square$

Lemma 7. *Let $\mathbf{A}$ be a finite group. Let S be a nonsingular, irreducible, quasi-projective scheme with a $\mathbb{C}^* \times \mathbf{A}$-action and a $\mathbb{C}^* \times \mathbf{A}$-fixed point $s \in S$. Let T_s denote the $\mathbb{C}^* \times \mathbf{A}$-representation on the tangent space at s. Suppose the $\mathbb{C}^*$-action is fully definite at s. Then, there is $\mathbb{C}^* \times \mathbf{A}$-equivariant isomorphism between an open set of (S, s) and $(T_s, 0)$.*

Proof. We note $\mathbb{C}^* \times \mathbf{A}$ is a linearly reductive group. By Theorem 2.4 of [Bi] for linearly reductive group actions, we may find a third nonsingular irreducible pointed space (Z, z) with a $\mathbb{C}^* \times \mathbf{A}$-action and equivariant, étale, morphisms:

$$\pi_1 : (Z, z) \to (S, s),$$

$$\pi_2 : (Z, z) \to (T_s, s).$$

In the proof of Theorem 2.5 of [Bi], such morphisms π_1 and π_2 are proven to be open immersions *by a study of only the $\mathbb{C}^*$-action.* Hence, the morphisms π_1 and π_2 are open immersions in our case. By the full definiteness of the $\mathbb{C}^*$-representation on T_s, the morphism π_2 is then an isomorphism. $\square$

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Pohang University of Science and Technology
E-mail address: `bumsig@postech.ac.kr`

California Institute of Technology
E-mail address: `rahulp@cco.caltech.edu`